Int. J. Heat Mass Transfer. Vol. 25, No. 4, pp. 587–589, 1982
Printed in Great Britain

0017 9310/82/040587-03 $03.00/0
Pergamon Press Ltd.

SHORTER COMMUNICATIONS

ON THE PERFORMANCE OF THE ENTHALPY METHOD

GRAHAM E. BELL
Department of Applied Mathematics, University of St. Andrews,
St. Andrews, Fife, Scotland

(*Received* 15 *September* 1981 *and in revised form* 15 *October* 1981)

NOMENCLATURE

A_n, Fourier coefficients;
C, heat capacity per unit volume;
H, enthalpy;
K, conductivity;
L, latent heat of freezing;
m, reference of mesh point undergoing transition;
t, time;
t', time measured from the moment point m becomes zero;
T, temperature;
x, depth from ground surface;
x_0, depth of frost.

Greek symbols

Δt, time increment;
Δx, space increment;
κ, thermal diffusivity.

Subscripts

f, frozen region;
u, unfrozen region.

INTRODUCTION

THE ENTHALPY method [1] is a simple and flexible technique for solving heat transfer problems involving either melting or freezing. Instead of working entirely in terms of the temperature of a material, an enthalpy function is defined which represents the total heat content per unit mass of the material. The advantage of such a reformulation is that the necessity to carefully track the location of the solid–liquid interface is removed and standard numerical techniques can be employed. For problems in which the thermal properties differ between the solid and liquid states the mesh point undergoing transition needs to be monitored. However, this is a trivial computational task compared with the otherwise time consuming chore of catering for different numerical approximations at each of the points adjacent to the interface.

Voller *et al.* [2] applied the enthalpy method to a test problem posed by Goodrich [3] and found that, although the predicted temperature distributions were reasonable, the predicted time history of a typical point was far from acceptable. The predicted temperature of a point displayed a pronounced step-like behaviour. They suggest a possible cause for this phenomenon and in a more recent paper [4] propose a procedure for rectifying the situation.

The purpose of this short communication is to explain the reason for the unusual behaviour of the numerical solution and why it is so exaggerated for Goodrich's test problem. This, in its turn, provides a justification for the remedial procedure proposed by Voller and colleagues. However, the degree of success they report will not necessarily be repeated when tackling other problems.

GOODRICH'S TEST PROBLEM

Consider the problem of the freezing of the ground, initially at 2°C, as a result of an instantaneous drop in surface temperature to −10°C. The conductivity, K, latent heat per unit volume, L, and heat capacity per unit volume, C, are taken as: $K_f = 2.25$ W/m°C; $C_f = 1.5$ MJ/m³ °C; $K_u = 1.75$ W/m°C; $C_u = 2.5$ MJ/m³ °C; $L = 100$ MJ/m³, where the subscripts denote the values in the frozen and unfrozen regions, respectively. The ground freezes at zero temperature and it is assumed that no volumetric change takes place. Within each region the usual heat conduction equation holds and at the interface, $x_0(t)$, between frozen and unfrozen ground the thermal balance

$$K_f\left(\frac{\partial T}{\partial x}\right)_{x_0^-} - K_u\left(\frac{\partial T}{\partial x}\right)_{x_0^+} = L\dot{x}_0$$

is maintained.

The equivalent enthalpy formulation is to consider the equation

$$\frac{\partial H}{\partial t} = K\frac{\partial^2 T}{\partial x^2}, \tag{1}$$

for all $x > 0$ where the enthalpy H and temperature T have the relationship

$$\begin{aligned} &T = (H - L)/C_u, \quad H > L; \\ &T = 0, \quad 0 \le H \le L; \\ &T = H/C_f, \quad H < 0. \end{aligned} \tag{2}$$

Replacing equation (1) by an explicity finite-difference approximation on a grid of size $\Delta x = 0.01$ m and $\Delta t = 30$ s, a numerical solution was computed from an initial temperature distribution corresponding to the exact solution [5] with $t = 7$ h. This starting condition was chosen to give a reasonable number of mesh points (i.e. 10) in the frozen phase initially, which avoids unnecessary numerical error. The numerical solution was allowed to progress until time $t = 27$ h.

Two aspects of the numerical solution are presented in Figs. 1 and 2. Figure 1 illustrates the predicted temperature distribution after 27 h and it is a fair guide to the actual temperature distribution (given in [5]). In Fig. 2 the temperature at a depth of 0.15 m is recorded. The step-like behaviour of the temperature is apparent. A carefully drawn graph also indicates that this phenomenon is not just confined to points in the frozen region. The existence of "steps" in the temperature history when unfrozen can be easily demonstrated with the aid of a difference table. However, for this example, the uneven fall in temperature in the unfrozen region is less dramatic.

A further scrutiny of the time history of the point 0.15 m reveals a number of interesting features. Firstly, the depth of each step reflects the size of the temperature gradient, which partly explains the different scale of effect in each region. However, a more fascinating observation is made when the length of each step is measured. The number of increments in

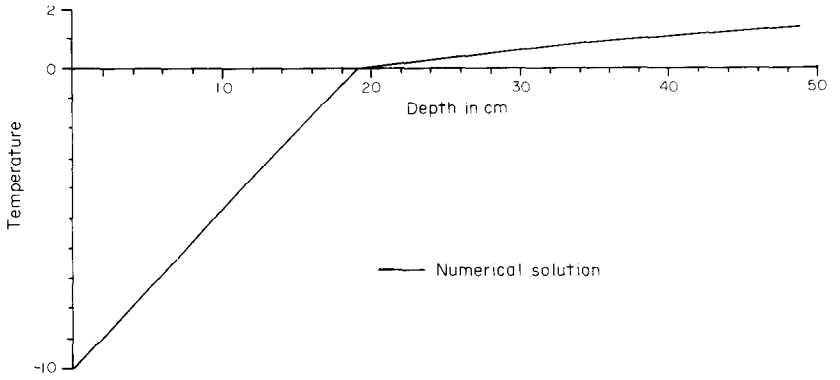

FIG. 1. Temperature distribution at $t = 27$ h.

Table 1

Step	Approx. length (no. of Δt's)	Interval (m)	Precise time taken to freeze interval (no. of Δt's)
0	257	[0.145, 0.155]	257.6
A	275	[0.155, 0.165]	274.8
B	293	[0.165, 0.175]	292.0
C	310	[0.175, 1.85]	309.2

Δt between consecutive jumps in temperature, indicated by the letters O, A, B and C in Fig. 2, are tabulated in Table 1. The precise time taken for frost to penetrate a distance Δx can be obtained from

$$x_0(t) = 0.508691(\kappa_f t)^{1/2}, \tag{3}$$

which is derived from the solution given by Carslaw and Jaeger [5]. The time, in terms of number of increments, that it takes to freeze successive Δx's of ground are also presented in Table 1. The close agreement between the step lengths and these times is remarkable.

Clearly, the temperature distribution throughout the medium remains static as the moving interface remains in the locality of a particular mesh point. A change in temperature only occurs when the interface moves from the locality of one mesh point to the next. Why does the numerical solution perform so well in some respects and so poorly in others? A review of Fig. 1 provides the answer. The temperature distribution in the frozen phase is effectively linear. To some extent this can also be said of the unfrozen region. However, for the moment consider the frozen region. The freezing is taking place very slowly compared to the time scale in which conduction is significant. The temperature within the frozen phase is quasi-steady. The numerical solution reflects this process very accurately.

Suppose the interval $[(m - \frac{1}{2})\Delta x, (m + \frac{1}{2})\Delta x]$ is undergoing transition from liquid to solid, the enthalpy at the point $m\Delta x$ lies in the interval $(0, L)$ and the temperature at $m\Delta x$ is taken to be zero. For the period of transition the process within $[0, m\Delta x]$ may be thought of as a separate boundary value problem and the numerical solution at the points within this range will mimic the behaviour of the expression

$$-10\left(1 - \frac{x}{m\Delta x}\right) + \sum_{n=1}' A_n \sin\frac{n\pi x}{m\Delta x} \exp\{-n^2\pi^2\kappa_f t'/m^2\Delta x^2\}, \tag{4}$$

where t' is measured from the moment the temperature at $x = m\Delta x$ becomes zero. It may easily be confirmed that the transient effects are indeed shortlived. The intervals over which transient conduction is significant correspond to the regions of rapid temperature change in Fig. 2. It is also informative to assess the magnitude of the exponent in

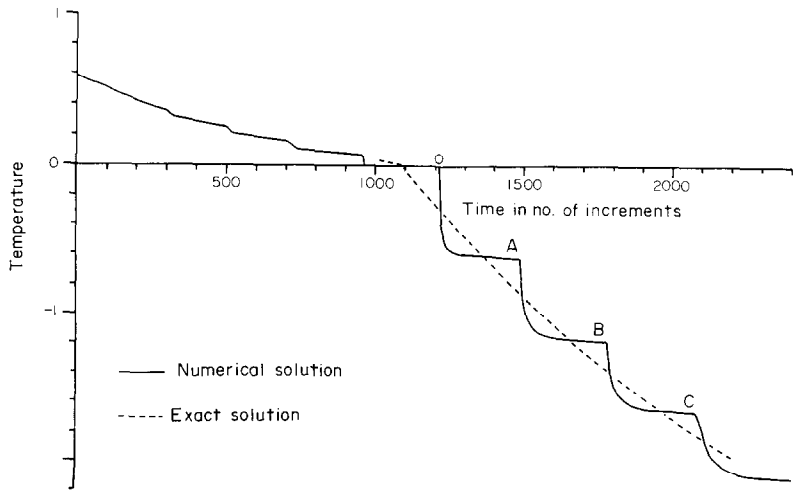

FIG. 2. Time history of the depth 15 cm.

expression (4) for the period of transition. From equation (3) the transition period is

$$2m(\Delta x)^2/(\kappa_f \times 0.258767) \text{ seconds}$$

and the exponent of the dominant transient term becomes

$$-2\pi^2/(m \times 0.258767). \quad (5)$$

As m increases, the above expression decreases and the time scale of the transient effects increases in comparison to the transition time. This too is portrayed by the numerical solution presented in Fig. 2. Although successive steps get longer, the periods of steady temperature get shorter.

In conclusion, the numerical solution, in its own way, gives a very accurate description of the melting process. The undesirable step-like behaviour produced by the enthalpy approach is merely a consequence of the quasi-steady nature of the temperature distribution.

In the unfrozen region a similar argument may be applied. The effect is considerably less dramatic as the form to which the solution tends during a transition period is itself non-steady (e.g. $\sim 2\,\text{erf}[(x - m\Delta x)/2(\kappa_u t')^{1/2}]$). It is only at mesh points soon to be frozen that any unevenness in the temperature fall is easily observed.

FINAL REMARKS

The behaviour described here will only occur when the heat capacity in the frozen region is very much less than the latent heat of freezing. However, this is frequently the case in industrial processes.

To overcome the uneven temperature fall Voller *et al.* [4] recommend a scheme in which linear interpolation is used to find the time at which the solid–liquid interface is at a mesh point. By using implicit finite-difference approximations they can allow the interface to move from one mesh point to the next in each time increment. For a time history of a point close to the surface this is ideal. As a consequence of (5) transient conduction is insignificant for a considerable period. The temperature profile in the frozen ground jumps from one equilibrium to the next and within each transition period the enthalpy at the transition point, $(m\Delta x)$, changes at an approximately constant rate. Linear interpolation is sufficient to determine the time at which the enthalpy takes the value $L/2$. When a point much further from ground level becomes frozen the time increments will be large and transient conduction will be more significant. This remedial scheme may be less successful.

The introduction of a small freezing range about 0°C appears an obvious alternative approach. The temperature at the transition point can then drift downwards during transition and prevent an equilibrium state being reached. However, Voller *et al.* [2] report little success with such techniques. There is scope for further research in this area.

Finally, it is worth noting that the numerical solution is sensitive to the initial enthalpy distribution. If the initial temperature is the sole criterion for determining the initial enthalpy, the solution will develop out of phase with the description presented here. All results presented in this short communication were based on an initial enthalpy distribution in which the enthalpy at the initial transition point was related to the known location of the solid–liquid interface in the manner described by Crank [6].

REFERENCES

1. M. E. Rose, A method for calculating solutions of parabolic equations with a free boundary, *Math. Comput.* **14**, 249–256 (1960).
2. V. R. Voller, M. Cross and P. G. Walton, Assessment of weak solution numerical techniques for solving Stefan problems. In *Numerical Methods in Thermal Problems* (Edited by R. W. Lewis and K. Morgan), pp. 172–181. Pineridge Press, Swansea, U.K. (1979).
3. L. E. Goodrich, Efficient numerical technique for one-dimensional thermal problems with phase change, *Int. J. Heat Mass Transfer* **21**, 615–621 (1978).
4. V. R. Voller and M. Cross, Accurate solutions of moving boundary problems using the enthalpy method, *Int. J. Heat Mass Transfer* **24**, 545–556 (1981).
5. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, 2nd edn., p. 285. Clarendon Press, Oxford (1959).
6. J. Crank, A nostalgic look at the mechanical solution of Stefan problems, *Bull. I.M.A.* **11**, 32–33 (1975).